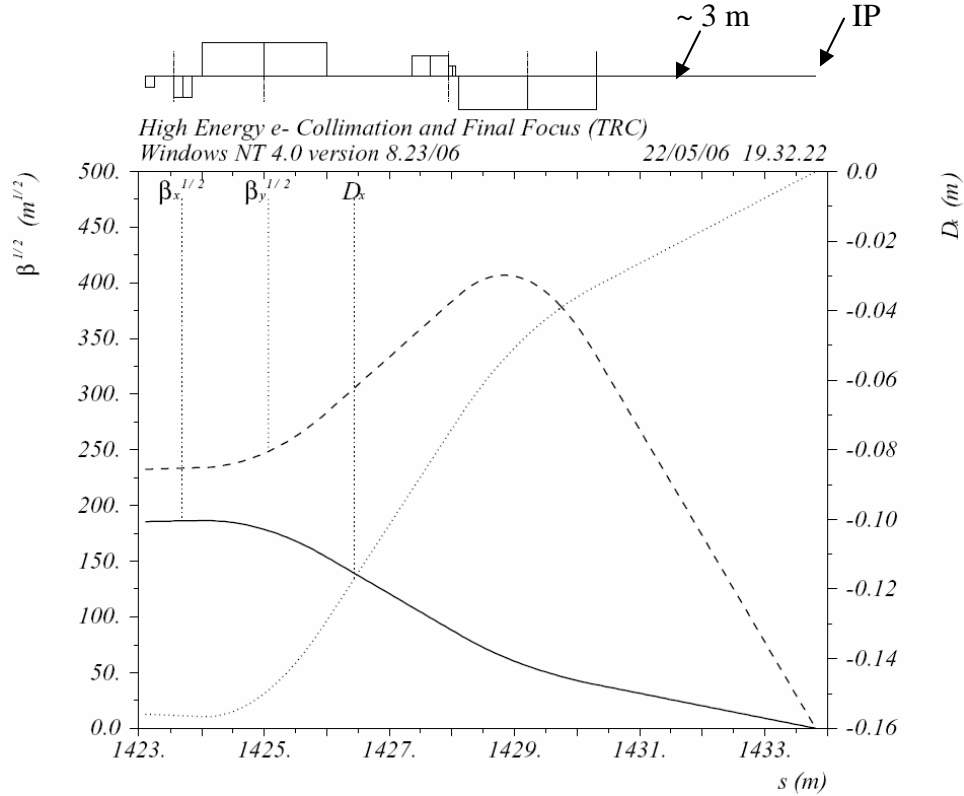


# The Final Three Meters in the ILC – Can the Beam Approach 5 nm?

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It is known that while the cost of the ILC is determined by the beam energy, its technical challenge comes from the luminosity. Compared with the SLC, the increase in beam energy is a factor of 5 (from 50 to 250 GeV), but in luminosity, a factor of 10,000 (from  $10^{30}$  to  $10^{34} \text{ cm}^{-2}\text{s}^{-1}$ ). To reach such a high luminosity, one must make the beam size very small at the IP ( $\sigma_y = 5 \text{ nm}$ ). The question is: *Can a dense charged particle beam be made so tiny?* This memo gives a quantitative analysis of the space charge force, which would ultimately determine the smallest beam size that could be obtained.

The ILC final focusing and beam envelope are shown in Figure 1. The vertical beam size  $\sigma_y$  would be reduced by a factor of 20,000 (from  $100 \text{ } \mu\text{m}$  to  $5 \text{ nm}$ ) in the final three meters before the beam reaches the IP.



**Figure 1:** ILC final focusing doublet. (Courtesy A. Seryi)

The electric and magnetic space charge forces of the electron (or positron) beam are strong. Due to opposite direction of the two forces, the total space charge force  $e \times [\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$  would have nearly perfect cancellation ( $1 - \beta^2 = 1/\gamma^2$ ) at high energy. However, this is only possible if the beam has zero emittance (i.e., no divergence). The ILC beam divergences are respectively,  $\sigma_x' = 31.2 \text{ } \mu\text{rad}$ ,  $\sigma_y' = 14.3 \text{ } \mu\text{rad}$ . Thus, for example, in the Y direction there would be  $1.43 \times 10^{-5}$  part of the Coulomb force ( $e \times \mathbf{E}$ ) that would not

be compensated. This residual force could still be large compared to the external magnetic focusing force. The following is a quantitative comparison.

We will use two known results:

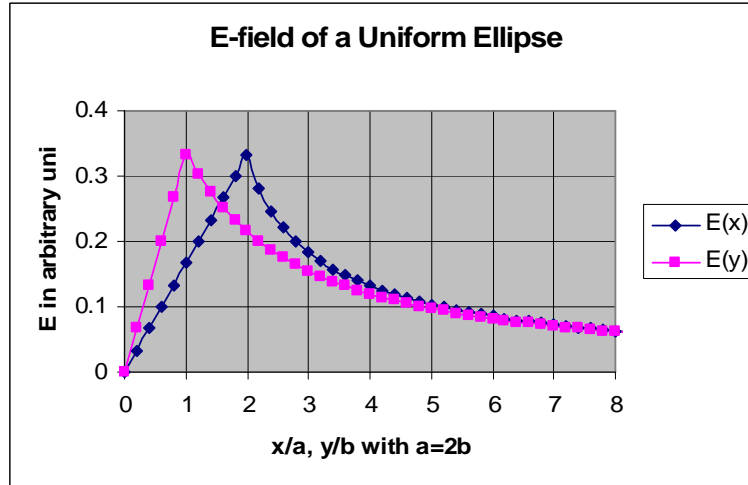
- 1) The space charge electric field of a 2-D ellipse and a 3-D ellipsoid with uniform charge distribution can be found in Refs. [1-6].
- 2) We are interested in beam envelope, which is determined by the second moment and not by the charge distribution [7]. Therefore, a 3-D ellipsoid beam with Gaussian distribution of rms size  $(\sigma_x, \sigma_y, \sigma_z)$  can be replaced by an ellipsoid with uniform distribution of size  $(5\sigma_x, 5\sigma_y, 5\sigma_z)$  since both have the same second moment.

Because the longitudinal size of the ILC beam is much larger than the transverse one, we may simplify our calculation to a 2-D ellipse, which has uniform charge distribution and the size of  $(4\sigma_x, 4\sigma_y)$ . (Note: This gives the same second moment as an elliptical Gaussian beam in 2-D.) Let  $a$  and  $b$  be the semi-axis of the ellipse. The electric field inside and outside is shown in Figure 2 and can be expressed as:

$$E_x = \frac{1}{\pi\epsilon_0} \frac{\lambda}{a(a+b)} x \quad x \leq a$$

$$E_x = \frac{1}{\pi\epsilon_0} \frac{\lambda}{x + (x^2 + b^2 - a^2)^{1/2}} \quad x \geq a \text{ and } y = 0$$

in which  $\lambda$  is the line density and  $\epsilon_0$  is the vacuum permittivity. Similar expressions are for  $E_y$  with  $a$  and  $b$  interchanged and  $x$  replaced by  $y$ . (When  $x \geq a$  and  $y \neq 0$ , the expression is more complicated but of no interest to this memo.)



**Figure 2:** Electric field of an ellipse with uniform charge distribution ( $a = 2b$ )

The field is linear inside the ellipse and reaches maximum on the boundary:

$$E_x(\text{max}) = E_y(\text{max}) = \frac{1}{\pi\epsilon_0} \frac{\lambda}{(a+b)}$$

By employing the following ILC parameters:

$$\begin{aligned}
N &= 2 \times 10^{10} \quad \text{per bunch} \\
\sigma_x &= 0.655 \quad \mu\text{m} \\
\sigma_y &= 5.72 \quad \text{nm} \\
\sigma_z &= 3 \times 10^{-4} \quad \text{m}
\end{aligned}$$

we get:

$$\begin{aligned}
\lambda &= N / [(2\pi)^{1/2} \sigma_z] = 2.7 \times 10^{13} \quad \text{particles/m} \\
a &= 4\sigma_x = 2.6 \times 10^{-6} \quad \text{m} \\
b &= 4\sigma_y = 2.3 \times 10^{-8} \quad \text{m}
\end{aligned}$$

The maximum space charge electric force on the boundary is:

$$F_x(\text{max}) = F_y(\text{max}) = 9.5 \times 10^{-9} \quad \text{N}$$

(It is interesting to note that this force is about 1/10 in magnitude of the bonding force between the electron and proton in a hydrogen atom.) As noted above, most of this force will be cancelled by the space charge magnetic force. The cancellation, however, won't be perfect due to finite beam emittance. The residual force can be estimated as:

$$F_y(\text{residual}) = F_x \times \sigma'_x = 3 \times 10^{-13} \quad \text{N}$$

$$F_x(\text{residual}) = F_y \times \sigma'_y = 1.4 \times 10^{-13} \quad \text{N}$$

This defocusing force should be compared to the external focusing force from the final doublet, of which the parameters are listed in Table 1.

**Table 1: ILC Final Doublet Parameters (2 mrad IR design)**

Magnet	Bore radius $R$ (mm)	Gradient $G$ (T/m)	Length $L$ (m)
QD	35	160	2.5
QF	10	68	2.0

The final vertical focusing mainly comes from QD. The beam envelope shown in Figure 1 assumes the final three meters are a free space (i.e., no focusing or defocusing force). However, this is not true and we already know the magnitude of the defocusing force. For the sake of comparison, we assume the external magnetic field was still there and can calculate the corresponding focusing force as follows:

$$B = G \times 2\sigma_y = 1.8 \times 10^{-6} \quad \text{T}$$

$$F_y(\text{focus}) = ev \times B = 1 \times 10^{-16} \quad \text{N}$$

It is seen that the defocusing force is three orders of magnitude larger than the focusing force. This brings up a problem – Even if the design emittance could be achieved (which is hard), we may still be unable to squeeze the beam to the design size at the IP because of the presence of a strong defocusing force in the final three meters.

#### References:

1. I.M. Kapchinsky and V.V. Vladimirovsky, Proc. Int. Conf. on High Energy Accelerators and Instrumentation, CERN, Sept. 1959, pp. 274-288.
2. L.C. Teng, ANL-Report, ANLAD-59 (1960).

3. P.M. Lapostolle, CERN Report AR/Int. SG/65-15 (1965).
4. K.R. Crandall, "TRACE 3-D Documentation," LA-UR-90-4146 (1990).
5. M.A. Furman, Am. J. Phys. **62**(12), 1994, pp. 1134-1140.
6. G. Parzen, BNL/SNS Technical Note No. 98 (2001).
7. F.J. Sacherer, PAC'71, Chicago, March 1971, pp. 1105-1107.

### Appendix 1 – ILC Beam Volume Density:

Another way to look at the difficulty for the ILC beam to reach the design size is to calculate the particle volume density. Table 2 lists the beam size and density for the ILC, SLC, FFTB and ATF, respectively.

**Table 2: Beam Size and Density**

	ILC (design)	SLC (achieved)	FFTB (achieved)	ATF (achieved)
Particles per bunch, $N$	$2 \times 10^{10}$	$3.5 \times 10^{10}$	$0.65 \times 10^{10}$	$3 \times 10^{10}$
$\sigma_x$ (mm)	$0.655 \times 10^{-3}$	$1.6 \times 10^{-3}$	$1 \times 10^{-3}$	$40 \times 10^{-3}$
$\sigma_y$ (mm)	$5.72 \times 10^{-6}$	$0.7 \times 10^{-3}$	$70 \times 10^{-6}$	$5 \times 10^{-3}$
$\sigma_z$ (mm)	0.3	1.2	1.2	9
Density, $N / \sigma_x \sigma_y \sigma_z$ ( $\text{mm}^{-3}$ )	$1.8 \times 10^{19}$	$2.6 \times 10^{16}$	$7.7 \times 10^{16}$	$1.7 \times 10^{13}$

The particle volume density in the ILC is so high that if we compare it with ordinary materials in the rest frame, it is three orders of magnitude higher than the ideal gas and even higher than some solid, e.g., potassium, as shown in Table 3.

**Table 3: Comparison of Particle Density**

	ILC (lab frame)	Ideal Gas (rest frame)	Potassium (rest frame)
Particle density ( $\text{mm}^{-3}$ )	$1.8 \times 10^{19}$	$2.7 \times 10^{16}$	$1.3 \times 10^{19}$

In reality, the situation is even more severe. While the particle distribution in gases and solids in the three dimensions is uniform, it is highly uneven in the ILC beam. The aspect ratio is  $Y: X: Z = 1: 115: 52450$ . The particles are much denser in the Y direction than in the other two directions. To estimate how dense it is, let us draw a box near the peak density point with the size  $Y \times X \times Z = 1 \text{ \AA} \times 115 \text{ \AA} \times 52450 \text{ \AA}$ . The number of particles in this box would be approximately  $1.1 \times 10^5$ , or 48 particles in each direction. In other words, there would be 48 electrons (or positrons) within  $1 \text{ \AA}$  (the size of an atom) in the Y direction. (In the SLC case, the same procedure leads to 0.47 electrons or positrons within  $1 \text{ \AA}$  in the Y direction, which is 100 times less than in the ILC.)

These densely populated particles result in an enormous space charge force, which, even after the relativistic cancellation, remains much larger than the external focusing force as demonstrated in the text.

## Appendix 2 – Modification of the Beam Envelope Equation:

The beam envelope, when the space charge is included, can be obtained by solving the following envelope equation [1,7]:

$$a_x'' + k_x(s)a_x - \frac{\varepsilon_x^2}{a_x^3} - \frac{2K}{a_x + a_y} = 0$$

$$a_y'' + k_y(s)a_y - \frac{\varepsilon_y^2}{a_y^3} - \frac{2K}{a_x + a_y} = 0$$

where  $a_x/a_y$  is the horizontal/vertical (h/v) beam envelope,  $k_x(s)/k_y(s)$  is the h/v focusing quadrupole strength,  $\varepsilon_x/\varepsilon_y$  is the h/v unnormalized emittance, and  $K$  is the dimensionless parameter (called the generalized perveance) describing the strength of the space charge:

$$K = \frac{I}{\beta\gamma^2 I_0}$$

in which  $I$  is the beam current and  $I_0$  is the Alfvén current:

$$I_0 = \frac{4\pi\varepsilon_0 mc^3}{e} \beta\gamma = 1.71 \times 10^4 \times \beta\gamma \text{ A for electrons}$$

A modification for the ILC beam in the final three meters is to replace  $1/\gamma^2$  in the expression of  $K$  by  $a_x'$  and  $a_y'$ , respectively:

$$K_x = \frac{a_y'}{\beta} \frac{I}{I_0}$$

$$K_y = \frac{a_x'}{\beta} \frac{I}{I_0}$$

In most cases when the space charge is of interest,  $\gamma$  is low and the beam divergence is much smaller than  $1/\gamma^2$ . As a result, the residual space charge force from the beam divergence can be ignored. However, the ILC is a special case. The beam divergence is much larger than  $1/\gamma^2$  and the modification is necessary.

For the ILC final focusing, we have:

$$k_x = 0.08 \text{ m}^{-2}$$

$$k_y = 0.19 \text{ m}^{-2}$$

$$\varepsilon_x = 2 \times 10^{-11} \text{ m-rad}$$

$$\varepsilon_y = 8 \times 10^{-14} \text{ m-rad}$$

The parameters  $K_x$  and  $K_y$  increase rapidly in the final three meters ( $\propto \beta_y^{-1/2}$  and  $\beta_x^{-1/2}$ , respectively) and approach the following values at the IP:

$$K_x = 2.2 \times 10^{-12}$$

$$K_y = 4.8 \times 10^{-12}$$

The modified envelope equation can be solved numerically. It is, however, beyond the scope of this memo.